Extreme residual dependence for random vectors and processes

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Abstract: A two-dimensional random vector in the domain of attraction of an extreme value distribution G is said to be asymtptotically independent (i.e. in the tail) if G is the product of its marginal distribution functions. Ledford and Tawn (1996) have discussed a form of residual dependence in this case. In this paper, we give a characterization of this phenomenon (see also Ramos and Ledford (2009)) and offer extensions to higher dimensional spaces and stochastic processes. Systemic risk in the banking system is treated in a similar framework.

AMS 2000 subject classifications: Primary 62G20, .
Keywords and phrases: asymptotic independence, extreme residual dependence, spectral measure.

1. Introduction

What is extreme residual dependence? Let us start by reminding the extreme value dependence in multivariate extreme value theory (EVT).

In a two-dimensional setup, we consider a random vector (X_1, X_2) with distribution function F. Denote its independent and identical distributed (i.i.d.) copies as $(X_1^{(1)}, X_2^{(1)}), (X_1^{(2)}, X_2^{(2)}), \cdots$. We take partial maxima for each marginal as $M_{i,n} = \max_{1 \le j \le n} X_i^{(j)}$, for $i = 1, 2$. Multivariate EVT assumes the following limit relation: there exist sequences of constants $a_{i,n} > 0$, $b_{i,n} \in \boldsymbol{R}$ for $i = 1, 2$ and a distribution function G with nondegenerate marginals, such that

$$\lim_{n\to\infty} P\left(\frac{M_{1,n} - b_{1,n}}{a_{1,n}} \le x_1, \frac{M_{2,n} - b_{2,n}}{a_{2,n}} \le x_2\right) = G(x_1, x_2), \qquad (1)$$

for all continuity points (x_1, x_2) of G. Then, the distribution function F is in the domain of attraction of a multivariate extreme value distribution G.

Equivalently, there exists real value functions $a_i(t) > 0$ and $b_i(t)$ for $i = 1, 2$ such that

$$\lim_{t\to\infty} tP\left(\frac{X_1 - b_1(t)}{a_1(t)} > x_1 \text{ or } \frac{X_2 - b_2(t)}{a_2(t)} > x_2\right) = -\log G(x_1, x_2), \tag{2}$$

for all continuity points (x_1, x_2) of G.

Let $F_i, i = 1, 2$ be the marginal distribution functions of F. Suppose F_i is a continuous distribution function. Multivariate EVT shows that, the necessary and sufficient condition for F being in the domain of attraction can be separated into two parts: each marginal distribution F_i belongs to the domain of attraction as in univariate EVT with extreme value index γ_i (for details on the univariate condition, see Theorem 1.1.6 in de Haan and Ferreira [2006]); the dependence structure satisfies: given any (x_1, x_2) for which $0 < G_0(x_1, x_2) < 1$,

$$\lim_{t\to\infty} t\left(1 - P\left(\frac{1}{1 - F_1(X_1)} \le tx_1, \frac{1}{1 - F_2(X_2)} \le tx_2\right)\right) = -\log G_0(x_1, x_2), \tag{3}$$

where $G_0(x_1, x_2) = G\left(\frac{x_1^{\gamma_1} - 1}{\gamma_1}, \frac{x_2^{\gamma_2} - 1}{\gamma_2}\right)$. Denote $\tilde{X}_i = \frac{1}{1 - F_i(X_i)}$ for $i = 1, 2$. Then the marginal distributions of $(\tilde{X}_1, \tilde{X}_2)$ are both standard Pareto distribution, i.e. $P(\tilde{X}_i > x) = 1/x$, for $x > 1$, which does not contain marginal information of (X_1, X_2). Hence, the relation (3) based on $(\tilde{X}_1, \tilde{X}_2)$ is a condition only on the extreme dependence of (X_1, X_2). Thus $G_0(x, y)$ characterizes the structure of *extreme value dependence*.

The extreme value dependence can be further decomposed as follows. Condition (3) holds if and only if there exists a measure ν on $\boldsymbol{R}_+^2$ such that for $x_1, x_2 > 0$

$$-\log G_0(x_1, x_2) = \nu\{(u, v) : u > x_1 \text{ or } v > x_2\}. \tag{4}$$

Then for any Borel set $A \subset \boldsymbol{R}_+^2$ with $\inf_{(x_1,x_2)\in A} x_1 \vee x_2 > 0$ and any $a > 0$,

$$\nu(aA) = a^{-1}\nu(A). \tag{5}$$

The measure ν is called the *exponent measure*. It has the following representation: there exists a probability measure H on $[0, 1]$ with mean $1/2$, such that

$$\nu\{(u, v) : u > x_1 \text{ or } v > x_2\} = 2\int_0^1 \frac{w}{x_1} \bigvee \frac{1 - w}{x_2} H(dw). \tag{6}$$

The measure H is called the *spectral measure*. The limiting distribution G_0 in (3) is determined by either ν or H.

Conversely, any exponent measures ν satisfying (5) or any probability measures H on $[0, 1]$ with mean $1/2$ occur as in (3)-(6). This can be seen by choosing

$$\begin{aligned} U_1 &= 2R\Theta, \\ U_2 &= 2R(1 - \Theta), \end{aligned} \tag{7}$$

where R and Θ are independent random variables with the distribution functions $P(R > r) = \frac{1}{r}$ for $r > 1$ and $P(\Theta \le w) = H(w)$ for $w \in [0, 1]$. To see this, firstly check that $F_{U_i}(x) := P(U_i \le x) = 1 - 1/x$, for $x > 2$, $i = 1, 2$. Thus $\frac{1}{1-F_{U_i}(x)} = x$, for $x > 2$. Next, check for any $x_1, x_2 > 0$,

$$\lim_{t\to\infty} tP\left(2R\Theta > tx_1 \text{ or } 2R(1-\Theta) > tx_2\right) = 2\int_0^1 \frac{w}{x_1} \bigvee \frac{1-w}{x_2} H(dw).$$

Then it is obvious that the random vector (U_1, U_2) belongs to the domain of attraction with spectral measure H. For details, see de Haan and Resnick [1977] and de Haan and Ferreira [2006].

A special case occurs when the spectral measure H is concentrated on the points $\{0\}$ and $\{1\}$ with measure $1/2$ each. In that case,

$$\nu\left\{(u,v) : u > x_1 \text{ or } \ v > x_2\right\} = \frac{1}{x_1} + \frac{1}{x_2} = \nu\left\{(u,v) : u > x_1\right\} + \nu\left\{(u,v) : v > x_2\right\},$$

for $x_1, x_2 > 0$, i.e. the exponent measure ν is concentrated on the coordinates. This additive property translates into a product property for the limit distribution function G_0 in (3), i.e. the distribution function G_0, hence G, is the product of two marginal distributions. This phenomenon is called *asymptotic independence* (cf. Geffroy [1958] and Sibuya [1960]). Furthermore, we get

$$\lim_{t\to\infty} tP\left(\tilde{X}_1 > tx_1 \text{ and } \tilde{X}_2 > tx_2\right) = \nu\left\{(u,v) : u > x_1 \text{ and } \ v > x_2\right\}) = 0,$$

for $x_1, x_2 > 0$. Hence no asymptotic information about sets of the form $\{(u,v) : u > x_1 \text{ and } v > x_2\}$ is obtained.

Ledford and Tawn in a series of papers Ledford and Tawn [1996, 1997, 1998, 2003] have filled in the gap by introducing an additional natural assumption that for some $0 < \eta < 1$ and all $x_1, x_2 > 0$, $P\left(\tilde{X}_1 > tx_1 \text{ and } \tilde{X}_2 > tx_2\right)$ is a regularly varying function with index $-1/\eta$. Such an extra assumption is closely related to the *second order condition* introduced by de Haan and Resnick [1993].

In multivariate EVT, the second order condition characterizes the speed of convergence in (2) as follows: there exists a non constant function $\psi(x_1, x_2)$ and a positive function $A(t) \to 0$ as $t \to \infty$, such that

$$\lim_{t\to\infty} \frac{tP\left(\frac{X_1 - b_1(t)}{a_1(t)} > x_1 \text{ or } \frac{X_2 - b_2(t)}{a_2(t)} > x_2\right) + \log G(x_1, x_2)}{A(t)} = \psi(x_1, x_2) < \infty, \tag{8}$$

holds locally uniformly for $0 < x_1, x_2 \le \infty$. This is a generalization of the second order condition in univariate case, see de Haan and Stadtmüller [1996]. The second order condition implies that $A(t)$ is a regularly varying function with index $\rho \le 0$. Similar to the first order case, the second order condition (8) implies that each marginal distribution F_i satisfies the univariate second order condition and, jointly, the dependence structure satisfies:

$$\lim_{t\to\infty} \frac{t\left(1 - P\left(\tilde{X}_1 \le tx_1, \tilde{X}_2 \le tx_2\right)\right) + \log G_0(x_1, x_2)}{A(t)} = \psi_0(x_1, x_2),$$

where $\psi_0(x_1,x_2)=\psi\left(\frac{x_1^{\gamma_1}-1}{\gamma_1},\frac{x_2^{\gamma_2}-1}{\gamma_2}\right)$. With the notation of the exponent measure ν, we have

$$\lim_{t\to\infty}\frac{tP\left(\tilde{X}_1>tx_1,\tilde{X}_2>tx_2\right)-\nu\left\{(u,v):u>x_1 \text{ and } \ v>x_2\right\}}{A(t)} \quad \text{exists.}$$

In the asymptotic independence case, this is simplified to

$$\lim_{t\to\infty}\frac{P\left(\tilde{X}_1>tx_1,\tilde{X}_2>tx_2\right)}{Q(t)} \quad \text{exists positive.} \tag{9}$$

where $Q(t):=A(t)/t$ is a positive regularly varying function with index $\rho-1\leq -1$. By defining $\eta=\frac{1}{1-\rho}$, the regularly varying index of $Q(t)$ is then $-1/\eta$. Hence, we get the same setup as Ledford and Tawn model.

We call that there is *extreme residual dependence* if (X_1,X_2) belongs to the domain of attraction of a bivariate extreme value distribution, they are asymptotic independent, and the condition (9) holds. The name residual dependence reflects that after eliminating the basic independence, there is still notable dependence for the residual part. The parameter η is called the *extreme residual coefficient*. An example is the bivariate normal distribution with correlation coefficient r less than one. In this case $\eta=(1+r)/2$.

There are quite a few papers on the "Ledford and Tawn model" in $\boldsymbol{R}^2$. For instance, extreme residual dependence has been discussed under the name "hidden regularly variation" in Resnick [2002], Maulik and Resnick [2004], Heffernan and Resnick [2005]. Draisma et al. [2004] study the estimation of the extreme residual coefficient. For application, Poon et al. [2004] apply both extreme value dependence and extreme residual dependence models in modeling financial returns from major stock indices. By estimating extreme residual coefficient, they identify asymptotic dependence and asymptotic independence among different pairs of stock indices.

For the characterization of extreme residual dependence, Ramos and Ledford [2009] consider bivariate regular variation and give a characterization of extreme residual dependence in two-dimensional case which relies on a spectral measure restricted by a normalization condition. Due to the restriction, in application, it is not straight forward how to verify the normalization condition, how to simulate random vectors exhibiting extreme residual dependence, and how to construct examples on extreme residual dependence from their characterization. Furthermore, although it is stated that the characterization in Ramos and Ledford [2009] can be extended to higher dimensional cases, the connection with higher dimensional EVT is not obvious.

We develop a theory characterizing the extreme residual dependence analogous to the traditional bivariate EVT as sketched in relations (4)-(6) above. Moreover, we generalize the Ledford and Tawn assumption into higher dimensional Euclidean spaces as well as in the context of stochastic processes. i.e. the infinite-dimensional case. The generalization is not trivial in the following sense. In $\boldsymbol{R}^2$, the Ledford and Tawn assumption on extreme residual dependence implies a unique asymptotic independence structure on ν or H: ν must concentrates on the coordinates axes while H must concentrate on two points $\{0\}$ and $\{1\}$. However, in higher dimensional case and in the stochastic process context, an analog of the Ledford and Tawn model does not correspond to a unique extreme value dependence structure exhibiting asymptotic independence. Instead, a variety of potential extreme value dependence structures may occur: asymptotic dependence may exist for some subsets of the components of the random vector but not for all components jointly. We shall derive a full characterization of the extreme residual dependence in this situation too.

2. Characterization of extreme residual dependence in $\boldsymbol{R}^2$

We start with condition (9). It is equivalent to

$$\lim_{t\to\infty} \frac{P(\tilde{X}_1^{1/\eta} > tx_1 \text{ and } \tilde{X}_2^{1/\eta} > tx_2)}{Q(t^\eta)} \quad \text{exists positive,}$$

for all $x_1, x_2 > 0$. Since $Q(t)$ is a regularly varying function with index $-1/\eta$, $Q(t^\eta)$ is a regularly varying function with index -1.

Similar to the two-dimensional EVT, there exists a measure ν^* on $(0, +\infty)^2$, finite on all sets $\{(u, v) : u > x_1 \text{ and } v > x_2\}$ for $x_1, x_2 > 0$, such that

$$\lim_{t\to\infty} \frac{P(\tilde{X}_1^{1/\eta} > tx_1 \text{ and } \tilde{X}_2^{1/\eta} > tx_2)}{Q(t^\eta)} = \nu^* \left\{(u, v) : u > x_1 \text{ and } v > x_2\right\}. \quad (10)$$

Clearly for $a > 0$ and a Borel set $B \subset (0, +\infty)^2$ that has a positive distance from both axes, i.e. $\inf_{(x_1,x_2)\in B} x_1 \wedge x_2 > 0$, we have that

$$\nu^*(aB) = a^{-1}\nu^*(B). \quad (11)$$

Consider the following one-to-one transformation $(0, +\infty)^2 \to (0, \infty)\times(0, \pi/2)$,

$$\begin{cases} r(u, v) &= \frac{1}{\frac{1}{u}+\frac{1}{v}}, \\ w(u, v) &= \frac{r}{u}. \end{cases} \quad (12)$$

Define for constants $r > 0$ and $w \in (0, 1)$ the set

$$B_{r,w} := \left\{(u, v) \in (0, +\infty)^2 : r(u, v) > r \text{ and } 0 < w(u, v) \leq w\right\}.$$

Notice that $\nu^*(B_{r,w}) < \infty$ for $r > 0$ and $w \in (0, 1)$, because $B_{r,w} \subset \{(u, v) : u > r \text{ and } v > r\}$. Since $B_{r,w} = rB_{1,w}$, we have that

$$\nu^*(B_{r,w}) = r^{-1}\nu^*(B_{1,w}).$$

Set $H^*(w) := \nu^*(B_{1,w})$, for $0 < w < 1$. Then H^* is a finite measure on $(0, 1)$. We show that all such measures occur. This is our main result in this section.

Theorem 2.1. *Let* (X_1, X_2) *be a random vector belonging to the domain of attraction of a two-dimensional extreme value distribution. Suppose* (X_1, X_2) *are asymptotically independent with extreme residual dependence structure given in* (10)*, and a extreme residual coefficient* η *lying in* $(1/2, 1)$*. Then, there exists a finite measure* H^* *on* $(0, 1)$ *such that*

$$\nu^* \left\{(u, v) : u > x_1 \text{ and } v > x_2\right\} = \int_0^1 \frac{1}{x_1 w} \bigwedge \frac{1}{x_2(1-w)} H^*(dw), \tag{13}$$

for $x_1, x_2 > 0$*. Conversely, for any finite measure* H^* *on* $(0, 1)$*, the right hand side of* (13) *is positive and finite. Moreover, there exists a random vector* (X_1, X_2) *exhibiting asymptotic independence in the two-dimensional EVT setup, and having extreme residual dependence structure given by* (10) *and* (13)*.*

Proof of Theorem 2.1

Firstly, with the construction of H^* above, we prove (13). Notice that for $r > 0$ and $0 < w < 1$, the inverse of the transformation (12) is

$$\begin{cases} u(r, w) &= \frac{r}{w}, \\ v(r, w) &= \frac{r}{1-w}. \end{cases} \tag{14}$$

The proof of (13) is then by calculation as follows,

$$\begin{aligned}
\nu^* \left\{(u, v) : u > x_1 \text{ and } v > x_2\right\} &= \nu^* \left\{(u, v) : \frac{r(u, v)}{w(u, v)} > x_1 \text{ and } \frac{r(u, v)}{1 - w(u, v)} > x_2\right\} \\
&= \nu^* \left\{(u, v) : r(u, v) > x_1 w(u, v) \vee x_2(1 - w(u, v))\right\} \\
&= \int_{(0,1)} H^*(dw) \int_{r > x_1 w \vee x_2(1-w)} \frac{1}{r^2} dr \\
&= \int_{(0,1)} \frac{1}{x_1 w} \bigwedge \frac{1}{x_2(1-w)} H^*(dw).
\end{aligned}$$

Conversely, starting with any given finite measure H^* on $(0, 1)$, the measure ν^* defined via (13) satisfies

$$\begin{aligned}
\nu^* \left\{(u, v) : u > x_1 \text{ and } v > x_2\right\} &= \int_{(0,1)} \frac{1}{x_1 w} \bigwedge \frac{1}{x_2(1-w)} H^*(dw) \\
&= \int_{(0,1)} \frac{1}{x_1 w \vee x_2(1-w)} H^*(dw) \\
&\leq \int_{(0,1)} \frac{2}{x_1 w + x_2(1-w)} H^*(dw) \\
&\leq \int_{(0,1)} \frac{2}{x_1 \wedge x_2} H^*(dw) \\
&= \frac{2}{x_1 \wedge x_2} H^*(0, 1) < \infty, \qquad (15)
\end{aligned}$$

for all $x_1, x_2 > 0$. Also, clearly ν^* satisfies the homogeneity property (11). Next, we prove that any finite measure H^* on $(0,1)$ may occur by constructing a suitable random vector (U_1, U_2) verifying all the requirements.

Our construction is separated into two steps. In the first step we construct a random vector (Z_1, Z_2) satisfying the residual dependence property.

Proposition 2.1. *Given any finite measure H^* on $(0,1)$, there exists a random vector (Z_1, Z_2) such that, for any $x_1, x_2 > 0$,*

$$\lim_{t\to\infty} tP(Z_1 > tx_1 \text{ and } Z_2 > tx_2) = \int_{(0,1)} \frac{1}{x_1 w} \bigwedge \frac{1}{x_2(1-w)} H^*(dw) \qquad (16)$$

$$\lim_{t\to\infty} t^\eta P(Z_i > t) = 0, \quad \text{for } i = 1, 2. \qquad (17)$$

Proof of Proposition 2.1

Consider two independent random variables R^* and Θ^* with distribution functions $P(R^* > x) = d/x$ for $x \geq d$ and $P(\Theta^* < w) = d^{-1} \int_{(0,w)} H^*(dw)$ for $0 < w \leq 1$, where $d = H^*(0,1)$. Since $1/2 < \eta < 1$, there exists a constant β such that $1 < \beta < 1/\eta$. Let

$$\begin{aligned} Z_1 &= \tfrac{R^*}{\Theta^*} \wedge (R^*)^\beta, \\ Z_2 &= \tfrac{R^*}{1-\Theta^*} \wedge (R^*)^\beta. \end{aligned} \qquad (18)$$

We first check relation (16) by calculation:

$$\begin{aligned}
&\lim_{t\to\infty} tP(Z_1 > tx_1 \text{ and } Z_2 > tx_2) \\
&= \lim_{t\to\infty} tP\left(\frac{R^*}{\Theta^*} > tx_1 \text{ and } \frac{R^*}{1-\Theta^*} > tx_2 \text{ and } (R^*)^\beta > t(x_1 \vee x_2)\right) \\
&= \lim_{t\to\infty} tE_{\Theta^*} \frac{d}{tx_1\Theta^* \vee tx_2(1-\Theta^*) \vee t^{1/\beta}(x_1 \vee x_2)^{1/\beta}} \\
&= \lim_{t\to\infty} \int_{(0,1)} \frac{1}{x_1 w \vee x_2(1-w) \vee t^{1/\beta - 1}(x_1 \vee x_2)^{1/\beta}} H^*(dw) \\
&= \int_{(0,1)} \frac{1}{x_1 w \vee x_2(1-w)} H^*(dw).
\end{aligned}$$

The final step comes from the Lebesgue dominated convergence theorem and the fact that the last integral is finite. We finish the proof of the proposition by verifying relation (17). Since $Z_i \leq (R^*)^\beta$,

$$\begin{aligned}
\limsup_{t\to\infty} t^\eta P(Z_i > t) &\leq \limsup_{t\to\infty} t^\eta P((R^*)^\beta > t) \\
&= \limsup_{t\to\infty} t^\eta \frac{d}{t^{1/\beta}} \\
&= 0.
\end{aligned}$$

Now we go back to the proof of Theorem 2.1. Let (Z_1, Z_2) be the random vector constructed in the proof of Proposition 2.1. Let (W_1, W_2) be independent random variables with distribution function $P(W_i > x) = 1/x$ for $x \geq 1$. Let (Z_1, Z_2) and (W_1, W_2) be independent. In the second step we assemble them by $U_i := W_i \vee Z_i^{\eta}$, $i = 1, 2$. We show that (U_1, U_2) satisfies all the requirements in Theorem 2.1.

Firstly, we study the marginal distributions of (U_1, U_2). Note that, for $i = 1, 2$ and $x > 0$

$$\begin{aligned}\lim_{t\to\infty} tP(U_i > tx) &= \lim_{t\to\infty} tP(W_i \vee Z_i^{\eta} > tx)\\ &= \lim_{t\to\infty} tP(W_i > tx) + tP(Z_i^{\eta} > tx) - tP(W_i > tx)P(Z_i^{\eta} > tx)\\ &= 1/x, \end{aligned} \tag{19}$$

because by (17), only the first term contributes. By denoting $F_{U_i}(x) := P(U_i \leq x)$, we have that $\frac{1}{1-F_{U_i}(t)} \sim t$ as $t \to \infty$.

Next, we check the extremal dependence structure as in (10) and (13). From the construction, we have that

$$\begin{aligned}&\lim_{t\to\infty} tP(U_1^{1/\eta} > tx_1 \text{ and } U_2^{1/\eta} > tx_2)\\ &= \lim_{t\to\infty} tP\left(\left\{W_1^{1/\eta} > tx_1 \text{ or } Z_1 > tx_1\right\} \text{ and } \left\{W_2^{1/\eta} > tx_2 \text{ or } Z_2 > tx_2\right\}\right)\\ &= \lim_{t\to\infty} tP((A_1 \cup B_1) \cap (A_2 \cup B_2)),\end{aligned}$$

where the sets are defined as $A_i := \left\{W_i^{1/\eta} > tx_i\right\}$ and $B_i := \{Z_i > tx_i\}$. From the expansion that

$$(A_1 \cup B_1) \cap (A_2 \cup B_2) = (A_1A_2) \cup (B_1A_2) \cup (A_1B_2) \cup (B_1B_2),$$

we get the lower and upper bound for $P((A_1 \cup B_1) \cap (A_2 \cup B_2))$ as

$$P(B_1B_2) \leq P((A_1\cup B_1)\cap(A_2\cup B_2)) \leq P(B_1B_2)+P(B_1A_2)+P(A_1B_2)+P(A_1A_2).$$

Proposition 2.1 shows that as $t \to \infty$, $tP(B_1B_2)$ converges as in (16). To prove that $tP((A_1 \cup B_1) \cap (A_2 \cup B_2))$ converges to the same limit, we only need to verify that as $t \to \infty$, $tP(B_1A_2)$, $tP(A_1B_2)$ and $tP(A_1A_2)$ converge to 0. Since the random vectors (W_1, W_2) and (Z_1, Z_2) are independent, considering (17) and the distribution function of W_i, we get that

$$\begin{aligned}\lim_{t\to\infty} tP(B_1A_2) &= \lim_{t\to\infty} tP(Z_1 > tx_1)P(W_2 > t^{\eta}x_2^{\eta})\\ &= \lim_{t\to\infty} t \cdot o(t^{-\eta})O(t^{-\eta}) = \lim_{t\to\infty} t^{1-2\eta}o(1) = 0,\end{aligned}$$

due to $1/2 < \eta < 1$. Similarly, we get that $\lim_{t\to\infty} tP(A_1B_2) = 0$. Furthermore, since W_1 and W_2 are independent, we get that

$$\lim_{t\to\infty} tP(A_1A_2) = \lim_{t\to\infty} t \cdot O(t^{-\eta})O(t^{-\eta}) = \lim_{t\to\infty} t^{1-2\eta}O(1) = 0.$$

Therefore, we proved that (cf. (16))

$$\lim_{t\to\infty} tP(U_1^{1/\eta} > tx_1 \text{ and } U_2^{1/\eta} > tx_2) = \lim_{t\to\infty} tP(Z_1 > tx_1 \text{ and } Z_2 > tx_2)$$
$$= \int_{(0,1)} \frac{1}{x_1 w} \bigwedge \frac{1}{x_2(1-w)} H^*(dw). \quad (20)$$

Together with the fact that $\frac{1}{1-F_{U_i}(t)} \sim t$ as $t \to \infty$, relation (20) is equivalent to the extreme residual dependence condition (9).

In the last step, we check that (U_1, U_2) belongs to the domain of attraction of a two-dimensional extreme value distribution with asymptotic independence. Write

$$tP(U_1 > tx_1 \text{ or } U_2 > tx_2) = tP(U_1 > tx_1) + tP(U_2 > tx_2)$$
$$- tP(U_1 > tx_1 \text{ and } U_2 > tx_2).$$

From (20) and (19), the extreme value dependence structure is obvious.

Remark 2.1. *Combining* (10) *and* (13), *we get that*

$$\lim_{t\to\infty} \frac{P(\tilde{X}_1 > tx_1, \tilde{X}_2 > tx_2)}{Q(t)} = \int_0^1 \frac{1}{x_1^{1/\eta} w} \bigwedge \frac{1}{x_2^{1/\eta}(1-w)} H^*(dw),$$

for all $x_1, x_2 > 0$, *which gives the limit in* (9). *The limit is a combination of the extreme residual coefficient* η *and the measure* H^*. *They can be independently chosen. It is different from the characterization in Ramos and Ledford [2009], which has a side condition on the two components.*

Remark 2.2. *Note that* ν^* *has the same homogeneity property as the exponent measure* ν, *however a* ν^* *measure is defined on Borel set* B *such that* $\inf_{(x_1,x_2)\in B} x_1 \wedge x_2 > 0$, *while a* ν *measure is defined on Borel set* B *such that* $\inf_{(x_1,x_2)\in B} x_1 \vee x_2 > 0$. *Hence, any exponent measure* ν *can act as a* ν^* *measure, but not vice versa. A* ν^* *measure can be extended to an exponent measure if and only if* $\nu^*\{(u,v) : u > x_1\}$ *and* $\nu^*\{(u,v) : v > x_2\}$ *are finite for all* $x_1, x_2 > 0$. *From* (13), *this is equivalent to the fact that* $\int_0^1 \frac{1}{w} H^*(dw)$ *and* $\int_0^1 \frac{1}{1-w} H^*(dw)$ *are finite. In that case, we say that* H^* *is a finite-type measure. Otherwise,* H^* *is called an infinite-type measure.*

Remark 2.3. *Although the original Ledford and Tawn model only requires* $0 < \eta < 1$, *we consider* $1/2 < \eta < 1$. *The condition is crucial for the proof. On the other hand, the case* $\eta > 1/2$ *is usually the one of interest in applications.*

Remark 2.4. *Theorem 2.1 gives the theoretical characterization for extreme residual dependence in two-dimensional situation. In practice, the constructive proof gives a method for simulating such a random vector when the measure* H^* *and the extreme residual coefficient* η *are known.*

3. Examples on extreme residual dependence in $\boldsymbol{R}^2$

We consider a few examples to demonstrate the application of the characterization.

Example 3.1. *Let H^* be concentrated on $1/2$ with measure 1. This is denoted as H_1 measure.*

With H_1 measure, we get that

$$\int_0^1 \frac{1}{x_1 w \vee x_2(1-w)} H_1(dw) = 2\left(\frac{1}{x_1} \wedge \frac{1}{x_2}\right).$$

Hence, the inequality (15) turns to be an equation in this case. To construct a random vector with such an extreme residual dependence structure, following the proof of Theorem 2.1, one may start by constructing (Z_1, Z_2) as in (7). Note that, with H_1 measure, $\Theta^* = 1/2$ is a constant. We get that $Z_1 = Z_2 = 2R^* \wedge (R^*)^\beta =: Z$. Hence, with (W_1, W_2) given as in the proof, by denoting $U_i = W_i \vee Z^\eta$ for $i = 1, 2$, (U_1, U_2) exhibits extreme residual dependence structure characterized by H_1 measure. Intuitively, this can be viewed as attaching a common factor Z^η on the asymptotically independent random vector (W_1, W_2). It is an analog of the completely tail dependent case as in bivariate EVT, i.e. the extreme residual part exhibits the strongest dependency.

Example 3.2. *Let H^* be the uniform probability distribution on $(0, 1)$. We denote $H_2(w) = w$.*

With the uniform distribution, $H_2(dw) = dw$, we get that

$$\begin{aligned}\int_0^1 \frac{1}{x_1 w \vee x_2(1-w)} H_2(dw) &= \left(\int_0^{\frac{x_2}{x_1+x_2}} + \int_{\frac{x_2}{x_1+x_2}}^1\right) \frac{1}{x_1 w \vee x_2(1-w)} dw \\ &= \frac{1}{x_2} \log \frac{x_1+x_2}{x_1} + \frac{1}{x_1} \log \frac{x_1+x_2}{x_2}.\end{aligned}$$

In Example 3.1, we have that

$$\int_0^1 \frac{1}{w} H_1(dw) = \int_0^1 \frac{1}{1-w} H_1(dw) = 2 < \infty.$$

In Example 3.2, we have that

$$\int_0^1 \frac{1}{w} H_2(dw) = \int_0^1 \frac{1}{1-w} H_2(dw) = +\infty.$$

Hence H_1 is a finite-type measure, while H_2 is an infinite-type measure.

The last example is based on the Beta distribution.

Example 3.3. *Consider a measure $H_3^{(\alpha,c)}$ defined as*

$$H_3^{(\alpha,c)}(dw) = c(w(1-w))^{\alpha-1} dw,$$

where $0 < \alpha \le +\infty$, and $c > 0$ is a normalization constant.

By taking proper normalization constant $c = c(\alpha)$ such that $H_3^{(\alpha,c(\alpha))}$ is a probability measure, it is the *Beta distribution* with parameter α, α.

The value of α shows how much the $H_3^{(\alpha,c)}$ measure concentrates on the central part of $(0, 1)$. Obviously, the uniform distribution in Example 3.2 corresponds to $\alpha = 1$ and $c = 1$. When normalizing the $H_3^{(\alpha,c)}$ measure to a probability measure, and taking $\alpha \to +\infty$, the limit is the H_1 measure in Example 3.1. Hence, the example covers the above two.

Moreover, it can be verified that $\int_0^1 \frac{1}{w} H_3^{(\alpha,c)}(dw)$ and $\int_0^1 \frac{1}{1-w} H_3^{(\alpha,c)}(dw)$ are finite if and only if $\alpha > 1$. Hence for $1 < \alpha \leq +\infty$, $H_3^{(\alpha,c)}$ is a finite-type measure, while for $0 < \alpha \leq 1$, $H_3^{(\alpha,c)}$ is an infinite-type measure.

Next, we calculate the explicit extreme residual dependence structure for a few specific values of α and c.

Take $\alpha = 1/2$ and $c = 1/2$. From

$$\int_{\frac{x_1}{x_1+x_2}}^1 \frac{(w(1-w))^{-1/2}}{w} dw = \sqrt{\frac{x_2}{x_1}},$$

we have that

$$\int_0^1 \frac{1}{x_1 w \vee x_2(1-w)} H_3^{(1/2,1/2)}(dw) = \frac{1}{\sqrt{x_1 x_2}}.$$

This is the extreme residual dependence structure in bivariate normal distribution. As shown in Draisma et al. [2004], Example 2.1, a bivariate normal distribution with mean 0, variance 1 and correlation coefficient $-1 < r < 1$ has extreme residual coefficient $\eta = \frac{1+r}{2}$, and the corresponding limit in (10) is $\sqrt{\frac{1}{x_1 x_2}}$. Hence the density of the H^* measure for the bivariate normal distribution is $\frac{dw}{2\sqrt{w(1-w)}}$.

Take $\alpha = 2$ and $c = 2$. From

$$\int_{\frac{x_1}{x_1+x_2}}^1 \frac{w(1-w)}{w} dw = \frac{x_2^2}{2(x_1+x_2)^2},$$

we have that

$$\int_0^1 \frac{1}{x_1 w \vee x_2(1-w)} H_3^{(2,2)}(dw) = \frac{1}{x_1 + x_2}.$$

Clearly, this is a finite-type example.

4. Extreme residual dependence in $\boldsymbol{R}^d$, $d \geq 3$

The Ledford and Tawn model on extreme residual dependence is originally defined in two-dimensional case. We generalize the definition of asymptotic independence and extreme residual dependence to random vectors in higher dimensional Euclidean space $\boldsymbol{R}^d$, $d \geq 3$.

In Section 4.1, we review the EVT in $\boldsymbol{R}^d$, $d \geq 3$ and generalize the definition of asymptotic independence in bivariate case to asymptotic joint independence in $\boldsymbol{R}^d$, $d \geq 3$. We provide the necessary and sufficient condition on the extreme value dependence structure corresponding to asymptotic joint independence.

In Section 4.2, we give the definition on extreme residual dependence in $\boldsymbol{R}^d$, $d \geq 3$, and show the difference from the bivariate case. Roughly speaking, in higher dimensional case ($d \geq 3$), the extreme residual dependence condition corresponds to asymptotic joint independence, but does not uniquely determine the extreme value dependence structure.

In Section 4.3,we present the full characterization of extreme dependence in $\boldsymbol{R}^d$, $d \geq 3$. Since the proof is parallel to that of the characterization in infinite-dimensional case, but simpler, we only present the result.

4.1. Extreme value dependence in $\boldsymbol{R}^d$, $d \geq 3$ and asymptotic joint independence

Suppose that a random vector $(X_1, X_2, \cdots, X_d)$ has joint distribution function $F(x_1, x_2, \cdots, x_d)$. Let $F_i(x_i)$, $i = 1, 2, \cdots, d$ be the marginal distribution functions of F. Suppose all F_i are continuous distribution functions. The distribution function F is in the domain of attraction of a $d-$dimensional extreme value distribution if and only if F_i belongs to the one-dimensional domain of attraction, and by denoting $\tilde{X}_i := \frac{1}{1-F_i(X_i)}$, there exists a positive measure ν such that

$$\lim_{t\to\infty} tP(\tilde{X}_1 > tx_1 \text{ or } \cdots \text{ or } \tilde{X}_d > tx_d) = \nu \left\{(t_1, \cdots, t_d) : t_1 > x_1 \text{ or } \cdots \text{ or } t_d > x_d\right\}. \tag{21}$$

Again, the measure ν is the *exponent measure*. It characterizes the limiting extreme value distribution.

For $d \geq 3$, the asymptotic independence can appear in several forms. Let us concentrate on the case $d = 3$. There are three levels of extreme value dependence:

1) The measure ν is concentrated on the three axes. Then

$$\lim_{t\to\infty} tP(\tilde{X}_i > tx_i \text{ and } \tilde{X}_j > tx_j) = 0,$$

for all $i \neq j$ and $x_i, x_j > 0$, and hence for all $x_1, x_2, x_3 > 0$

$$\lim_{t\to\infty} tP(\tilde{X}_1 > tx_1 \text{ and } \tilde{X}_2 > tx_2 \text{ and } \tilde{X}_3 > tx_3) = 0. \tag{22}$$

2) The measure ν is concentrated on the planes $\{(t_1, t_2, t_3) : t_i = 0\}$ for $i = 1, 2, 3$, but not only on the axes. Then one or more pairs (X_i, X_j) are not asymptotically independent, but (22) still holds. For the other pairs, theory of Section 2 may apply.

3) The measure ν assigns some positive measure on the open area

$$\{(t_1, t_2, t_3) : t_1 > 0, t_2 > 0, t_3 > 0\}.$$

Then (22) does not hold. All pairs (X_i, X_j) are asymptotically dependent.

Hence, (22) should be considered in defining asymptotic independence in $\boldsymbol{R}^3$. For the general case $d \geq 3$, we say that the random vector $(X_1, \cdots, X_d)$ is *asymptotically jointly independent* if and only if

$$\lim_{t\to\infty} tP(\tilde{X}_1 > tx_1 \text{ and } \cdots \text{ and } \tilde{X}_d > tx_d) = 0, \tag{23}$$

or equivalently

$$\nu\left\{(t_1, \cdots, t_d) : t_1 > x_1 \text{ and } \cdots \text{ and } t_d > x_d\right\} = 0,$$

for any $x_1, x_2, \cdots, x_d > 0$.

From the discussion on the case $d = 3$, we observe that a 3-d random vector in the domain of attraction is asymptotically jointly independent if and only if

$$\nu\left\{(t_1, t_2, t_3) : t_1 > 0, t_2 > 0, t_3 > 0\right\} = 0.$$

This can be generalized into the general case $d \geq 3$ as follows.

Theorem 4.1. *Suppose that a random vector $(X_1, \cdots, X_d) \in \boldsymbol{R}^d$ belongs to the domain of attraction of a $d-$dimensional extreme value distribution. It is asymptotically jointly independent as defined in* (23) *if and only if*

$$\nu\left\{(t_1, t_2, \cdots, t_d) : t_1 > 0 \text{ and } \cdots \text{ and } t_d > 0\right\} = 0. \tag{24}$$

4.2. Extreme residual dependence in $\boldsymbol{R}^d$, $d \geq 3$

In the setup of Section 4.1, if we have the asymptotically jointly independent case, i.e. (23) holds, as an analog of the bivariate case, we can define extreme residual dependence in $\boldsymbol{R}^d$ as follows. If there exists a regularly varying function $Q(t)$ with index $-1/\eta$ such that,

$$\lim_{t\to\infty} \frac{P(\tilde{X}_1 > tx_1 \text{ and } \cdots \text{ and } \tilde{X}_d > tx_d)}{Q(t)} \text{ exists positive,} \tag{25}$$

for any $x_1, x_2, \cdots, x_d > 0$, we say that there is *extreme residual dependence* with *extreme residual coefficient* η.

As shown in the proof of Theorem 2.1, in two-dimensional case, the combination of the extreme residual dependence condition (10) and the property on marginal distribution functions of $\tilde{X}_i$, $i = 1, 2$, automatically implies the EVT setup, with a unique spectral measure ν: ν must concentrate its measure on two axes, i.e. the extreme value dependence structure under asymptotic independence is unique. However, in the following example in case $d = 3$, we show that a given extreme residual dependence structure may correspond to different extreme value dependence structures.

Let (E_1, E_2) be an asymptotically dependent random vector with standard Pareto marginals and exponent measure ν_0, i.e. $P(E_i > x) = 1/x$, for $x > 1$, $i = 1, 2$, and

$$\lim_{t\to\infty} tP(E_1 > tx_1 \text{ or } \; E_2 > tx_2) = \nu_0\left\{(u, v) : u > x_1 \text{ or } v > x_2\right\} > 0,$$

for all $x_1, x_2 > 0$. Let E_3, E_4 be two independent standard Pareto distributed random variables. Moreover, they are independent from (E_1, E_2). We construct a 3-dimensional random vector (V_1, V_2, V_3) as

$$\begin{aligned} V_1 &= \max(E_1, E_4^{2/3}) \\ V_2 &= \max(E_2, E_4^{2/3}) \\ V_3 &= \max(E_3, E_4^{2/3}). \end{aligned}$$

It is not difficult to verify that all marginal distributions of (V_1, V_2, V_3) satisfy $1 - F_{V_i}(t) := P(V_i > t) \sim 1/t$ as $t \to \infty$, i.e. $\frac{1}{1-F_{V_i}(t)} \sim t$, for $i = 1, 2, 3$. Moreover, notice that

$$\begin{aligned} &\lim_{t\to\infty} tP(V_1 > tx_1 \text{ or } V_2 > tx_2 \text{ or } V_3 > tx_3) \\ =&\lim_{t\to\infty} tP(E_1 > tx_1 \text{ or } E_2 > tx_2 \text{ or } E_3 > tx_3 \text{ or } E_4 > (t\max(x_1,x_2,x_3)^{3/2})) \\ =&\lim_{t\to\infty} tP(E_1 > tx_1 \text{ or } E_2 > tx_2) + tP(E_3 > tx_3) + tP(E_4 > (t\max(x_1,x_2,x_3)^{3/2})) \\ =&\nu_0\left\{(u,v) : u > x_1 \text{ or } v > x_2\right\} + \frac{1}{x_3}, \end{aligned}$$

and

$$\begin{aligned} &\lim_{t\to\infty} t^{3/2}P(V_1 > tx_1 \text{ and } V_2 > tx_2 \text{ and } V_3 > tx_3) \\ =&\lim_{t\to\infty} t^{3/2}P\left(\{E_1 > tx_1 \text{ and } E_2 > tx_2 \text{ and } E_3 > tx_3\} \cup \left\{E_4 > (t\max(x_1,x_2,x_3)^{3/2})\right\}\right) \\ =&\lim_{t\to\infty} t^{3/2}P(E_1 > tx_1 \text{ and } E_2 > tx_2)P(E_3 > tx_3) + t^{3/2}P(E_4 > (t\max(x_1,x_2,x_3)^{3/2})) \\ =&\lim_{t\to\infty} t^{3/2}\frac{\nu_0\left\{(u,v) : u > x_1 \text{ and } v > x_2\right\}}{t}\frac{1}{tx_3} + \frac{1}{\max(x_1,x_2,x_3)^{3/2}} \\ =&\frac{1}{\max(x_1,x_2,x_3)^{3/2}}. \end{aligned}$$

Hence, (V_1, V_2, V_3) is asymptotically jointly independent in $\boldsymbol{R}^3$, with extreme residual coefficient $2/3$.

On one hand, the extreme value dependence structure of (V_1, V_2, V_3) depends on the choice of ν_0; on the other hand, the extreme residual dependence structure of (V_1, V_2, V_3) does not depend on the choice of ν_0. Hence, by varying the measure ν_0, we could have different random vectors with the same extreme residual dependence structure but different extreme value dependence structures.

We remark that, in the above example V_1 and V_2 are asymptotically dependent, which means that being a higher dimensional asymptotically jointly independent random vector does not rule out the possibility of asymptotic dependence within the subset of its components.

4.3. Characterization of extreme residual dependence in $\boldsymbol{R}^d$, $d \geq 3$

Similar to the two-dimensional case, in $\boldsymbol{R}^d$, $d \geq 3$, if the extreme residual dependence condition (25) holds, there exists a measure ν^* on $(0,\infty)^d$, finite on all sets

$$\{(t_1, \cdots, t_d) : t_1 > x_1 \text{ and } \cdots \text{ and } t_d > x_d\}$$

for $x_1, \cdots, x_d > 0$ such that

$$\lim_{t\to\infty} \frac{P(\tilde{X}_1^{1/\eta} > tx_1, \cdots, \tilde{X}_d^{1/\eta} > tx_d)}{Q(t^\eta)} = \nu^* \{(t_1, \cdots, t_d) : t_1 > x_1, \cdots, t_d > x_d\}. \tag{26}$$

Clearly, for all $a > 0$ and Borel set $B \subset (0,\infty)^d$ such that $\inf_{(x_1,\cdots,x_d)\in B} \bigvee_{i=1}^d x_i > 0$, we have

$$\nu^*(aB) = a^{-1}\nu^*(B). \tag{27}$$

It is a homogeneity property analogous to (11) in the two-dimensional case. The measure ν^* characterizes the extreme residual dependence structure.

As shown in Theorem 4.1, given the extreme residual dependence structure, the extreme value dependence structure characterized by the exponent measure ν must satisfy the condition (24). That corresponds to a variety of potential extreme value dependence structure. It is thus a question whether all possible extreme value dependence and extreme residual dependence structure may occur. The following theorem gives positive answer to this question which closes the discussion on characterization of extreme residual dependence in $\boldsymbol{R}^d$, $d \geq 3$.

Theorem 4.2. *Let $(X_1, X_2, \cdots, X_d)$ be a random vector belonging to the domain of attraction of a $d-$dimensional extreme value distribution with continuous marginal distributions and the extreme value dependence is characterized by an exponent measure ν as in (21). Suppose $(X_1, X_2, \cdots, X_d)$ are asymptotically jointly independent with extreme residual dependence structure characterized by a ν^* measure as in (26), and a extreme residual coefficient η lying in $(1/2, 1)$. Then ν must satisfy (24), and ν^* must be finite on all sets*

$$A_{(x_1,\cdots,x_d)} := \{(t_1, \cdots, t_d) : t_1 > x_1 \text{ and } \cdots \text{ and } t_d > x_d\}$$

for $x_1, \cdots, x_d > 0$ and satisfy the homogeneity condition (27).

Conversely, given any exponent measure ν satisfying (24), any ν^* measure finite on all sets $A_{(x_1,\cdots,x_d)}$ with the homogeneity condition (27) holds and any η lying in $(1/2, 1)$, there exists a random vector $(X_1, X_2, \cdots, X_d)$ belonging to the domain of attraction of a $d-$dimensional extreme value distribution, with exponent measure ν, exhibiting asymptotic joint independence, and having extreme residual dependence structure characterized by ν^* and extreme residual coefficient η.

We skip the proof of Theorem 4.2, because it follows similar lines as the proof for the infinite-dimensional case in Section 5, but simpler.

Theorem 4.2 shows that for a random vector with extreme residual dependence, the extreme value dependence structure, the extreme residual dependence structure and the extreme residual coefficient are independent. Any combination of these three components may occur.

In case $d = 3$, when (X_1, X_2, X_3) exhibits extreme residual dependence with extreme residual coefficient η and measure ν^*, a lower-dimensional subsets of the random vector, for instance (X_1, X_2), may be asymptotically independent or asymptotically dependent. In case (X_1, X_2) are asymptotically independent, they may have an extreme residual dependence structure characterized by a measure $\nu^*_{1,2}$ and an extreme residual coefficient $\eta_{1,2}$. It is obvious that, the extreme residual coefficient must satisfy the inequality that $\eta \leq \eta_{1,2} < 1$. Similar to the discussion above, any ν^*, $\nu^*_{1,2}$, and $\eta \leq \eta_{1,2} < 1$ can occur. This can be generalized to higher dimensional case, $d \geq 3$.

5. Extreme residual dependence for stochastic processes

We consider stochastic processes defined on the unit interval $[0, 1]$. Let $\{X(s)\}_{s\in[0,1]}$ be a continuous sample path stochastic process, i.e. an random element in the space $C[0, 1]$ equipped with L_∞ norm $|f|_\infty := \sup_{s\in[0,1]} f(s)$. In this section, we generalize the definition of asymptotic joint independence and extreme residual dependence to continuous sample path stochastic processes.

In Section 5.1, we review the EVT in $C[0, 1]$ and study the concept asymptotic joint independence in infinite-dimensional case. In Section 5.2, we generalize the definition of extreme residual dependence into $C[0, 1]$, and give its full characterization.

5.1. *Extreme value dependence and asymptotic joint independence in $C[0,1]$*

Suppose a continuous sample path stochastic process $\{X(s)\}_{s\in[0,1]}$ belongs to the domain of attraction of a max-stable process (for details, see de Haan and Ferreira [2006], Chapter 9). Supopse the marginal distribution function of $X(s)$, $F_s(x)$, is a continuous distribution function. The domain of attraction condition is equivalent to the combination of two conditions: firstly, $F_s(x)$ belongs to the one-dimensional domain of attraction with extreme value index $\gamma(s)$, where $\gamma(s)$ is a continuous real value function on $[0, 1]$; secondly, by denoting $\left\{\tilde{X}(s)\right\}_{s\in[0,1]} := \left\{\frac{1}{1-F_s(X(s))}\right\}_{s\in[0,1]}$, $\left\{\tilde{X}(s)\right\}_{s\in[0,1]}$ is also a continuous sample path stochastic process belonging to the same domain of attraction. EVT in $C[0, 1]$ shows that there must exist a finite measure ν on $\bar{C}^+[0, 1] := \{f \in C[0, 1] : f \geq 0\}$ such that for all Borel set $A \subset \bar{C}^+[0, 1]$ with $\inf\{|f|_\infty : f \in A\} > 0$ and $\nu(\partial A) = 0$,

$$\lim_{t\to\infty} tP(\tilde{X} \in tA) = \nu(A). \tag{28}$$

A corresponding spectral measure ρ is given in Giné et al. [1990] (cf. Theorem 9.4.1 in de Haan and Ferreira [2006]). For any $f \in \bar{C}^+[0, 1]$ with $|f|_\infty > 0$, we write $r := |f|_\infty$ and

$$f_1(s) := \frac{f(s)}{|f|_\infty} \in \bar{C}_1^+[0,1] := \{f \in C[0,1] : f \geq 0, |f|_\infty = 1\}.$$

Then f is decomposed into $(r, f_1) \in (0,\infty) \times \bar{C}_1^+[0,1]$. With such decomposition, the exponent measure ν is decomposed into a product measure as follows. There exists a finite measure ρ on $\bar{C}_1^+[0,1]$ with

$$\int_{\bar{C}_1^+[0,1]} f_1(s) d\rho(f_1) = 1, \tag{29}$$

for any $s \in [0,1]$, such that

$$\nu(A) = \int \int_{rf_1 \in A} \frac{dr}{r^2} d\rho(f_1). \tag{30}$$

Again, the measures ν and ρ are called the *exponent measure* and the *spectral measure*, respectively. Either of them characterizes the extreme value dependence for the stochastic process $\left\{\tilde{X}(s)\right\}_{s\in[0,1]}$, hence $\{X(s)\}_{s\in[0,1]}$.

We introduce the notation $f > g$ (or $f \geq g$) to indicate that two continuous function $f(s)$ and $g(s)$ defined on $s \in [0,1]$ satisfy that $f(s) > g(s)$ (or $f(s) \geq g(s)$) for all $s \in [0,1]$. Then, the extreme value dependence structure is equivalent to that, for any $g \in C^+[0,1] := \{f \in C[0,1] : f > 0\}$,

$$\lim_{t\to\infty} tP(\left\{\tilde{X} \leq tg\right\}^c) = \nu(\left\{f \in \bar{C}^+[0,1] : f \leq g\right\}^c) = \int_{\bar{C}_1^+[0,1]} \sup_{s\in[0,1]} \frac{f_1}{g} d\rho(f_1) > 0. \tag{31}$$

Similar to the definition on asymptotic joint independence in $\boldsymbol{R}^d$, $d \geq 3$ as in (23), we say that the stochastic process $\{X(s)\}_{s\in[0,1]}$ is *asymptotically jointly independent* if and only if for any $g \in C^+[0,1]$, the following relation holds,

$$\lim_{t\to\infty} tP(\left\{\tilde{X} > tg\right\}) = \nu(\left\{f \in \bar{C}^+[0,1] : f > g\right\}) = 0, \tag{32}$$

i.e. no information on the sets of the type $\left\{f \in \bar{C}^+[0,1] : f > g\right\}$ is available.

The following theorem provide the necessary and sufficient condition on the spectral measure which corresponds to asymptotic joint independence.

Theorem 5.1. *Suppose that a continuous sample path stochastic process $\{X(s)\}_{s\in[0,1]}$ belongs to the domain of attraction of some max-stable process with spectral measure ρ. The process is asymptotically jointly independent if and only if its spectral measure ρ satisfies*

$$\rho\left\{f_1 \in \bar{C}_1^+[0,1] : \inf f_1(s) > 0\right\} = 0. \tag{33}$$

Proof of Theorem 5.1
Denote $T := \left\{f_1 \in \bar{C}_1^+[0,1] : \inf f_1(s) > 0\right\}$ and $T_n := \left\{f \in \bar{C}_1^+[0,1] : \inf f(s) > 1/n\right\}$. Notice that that $T_1 \subset T_2 \subset \cdots \subset T$ and $\cup_{n=1}^\infty T_n = T$. The theorem follows.

Remark 5.1. *It is not difficult to verify that the condition* (33) *is equivalent to*

$$\int_{\bar{C}_1^+[0,1]} \inf f_1(s) d\rho(f_1) = 0.$$

Note (cf. Ferreira et al. [2009]) that

$$\int_{\bar{C}_1^+[0,1]} \inf f_1(s) d\rho(f_1) = \lim_{\gamma \downarrow -\infty} \theta_\gamma,$$

where

$$\theta_\gamma := \int_{\bar{C}_1^+[0,1]} \left(\int_{s\in[0,1]} f_1^\gamma(s) ds \right)^{1/\gamma} d\rho(f_1),$$

is the so-called areal coefficient in Coles and Tawn [1996].

5.2. Extreme residual dependence in $C[0,1]$

We generalize the definition of extreme residual dependence for stochastic processes as follows. Suppose that a continuous sample path stochastic process $\{X(s)\}_{s\in[0,1]}$ belonging to the domain of attraction of some max-stable process is asymptotically jointly independent, i.e. (32) holds. Analogous to (9), if there exists a regularly varying function $Q(t)$ with index $-1/\eta$ for some $0 < \eta < 1$ such that

$$\lim_{t\to\infty} \frac{P(\left\{ \tilde{X} > tg \right\})}{Q(t)} \text{ exists positive,} \tag{34}$$

for any $g \in C^+[0,1]$, we say that there is *extreme residual dependence* with *extreme residual coefficient* η.

From the definition of extreme residual dependence in (34), we get that

$$\lim_{t\to\infty} \frac{P\left(\left\{ \tilde{X}^{1/\eta} > tg \right\}\right)}{Q(t^\eta)} \text{ exists positive.}$$

Hence, there must exist a measure ν^* on $C^+[0,1]$, finite on sets of the type $A_g^* := \{f \in C[0,1] : f > g\}$ for any $g \in C^+[0,1]$, such that

$$\lim_{t\to\infty} \frac{P\left(\left\{ \tilde{X}^{1/\eta} > tg \right\}\right)}{Q(t^\eta)} = \nu^*(A_g^*). \tag{35}$$

For any $a > 0$, it can be verified that

$$\nu^*(A_{ag}^*) = a^{-1}\nu^*(A_g^*).$$

Hence, we get that

$$\nu^*(aA) = a^{-1}\nu^*(A), \tag{36}$$

for all A in the $\sigma-$field generated by A_g^*.

To get a spectral decomposition of ν^*, we consider the following transformation. For any $f \in C^+[0,1]$, since f is a continuous function, we have $m(f) := \inf_{s\in[0,1]} f(s) > 0$. Moreover, by denoting $f_2(s) := \frac{m}{f(s)}$, we have that $f_2 \in C_1^+[0,1] := \{f \in C^+[0,1] : |f|_\infty = 1\}$. Thus, any $f \in C^+[0,1]$ is decomposed into $(m, f_2) \in (0,+\infty) \times C_1^+[0,1]$. For any Borel set $B \subset C_1^+[0,1]$ and any $r > 0$, since

$$\nu^* \{f : m(f) > r, f_2 \in B\} = \frac{1}{r}\nu^* \{f : m(f) > 1, f_2 \in B\}.$$

By denoting $\Psi^*(B) := \nu^* \{f : m(f) > 1, f_2 \in B\}$, we decompose the measure ν^* into a product measure on $(0,+\infty) \times C_1^+[0,1]$. The measure ν^* and Ψ^* are connected by

$$\nu^*(A) = \int \int_{\frac{m}{f_2} \in A} \frac{dm}{m^2} d\Psi^*(f_2). \tag{37}$$

Note that

$$\Psi^*(C_1^+[0,1]) = \nu^* \{f : m(f) > 1\} = \nu^* \{f : f > 1\} < \infty.$$

We have that Ψ^* is a finite measure on $C_1^+[0,1]$.

Conversely, taking any finite measure Ψ^* on $C_1^+[0,1]$, we construct ν^* as above. Then,

$$\begin{aligned}
\nu^*(A_g^*) &= \nu^* \{f \in C[0,1] : f > g\} \\
&= \nu^* \left\{f \in C[0,1] : \frac{m}{f_2} > g\right\} \\
&= \nu^* \{f \in C[0,1] : m > \sup(f_2 g)\} \\
&= \int_{C_1^+[0,1]} \frac{1}{\sup(f_2 g)} d\Psi^*(f_2) \\
&\le \int_{C_1^+[0,1]} \frac{1}{\sup f_2 \cdot \inf g} d\Psi^*(f_2) \\
&= \frac{1}{\inf g} \Psi^*(C_1^+[0,1]).
\end{aligned}$$

Hence, for all continuous function $g > 0$, $\nu^*(A_g^*) < \infty$. Moreover, it is obvious that the constructed ν^* measure satisfies the homogeneity condition (36).

To summarize, we have shown that the extreme residual dependence condition can be characterized by either a measure ν^* on $C^+[0,1]$ or a measure Ψ^* on $C_1^+[0,1]$. The former must be finite on all sets of the type A_g^* and satisfying (36), while the latter must be a finite measure on $C_1^+[0,1]$.

As shown in Theorem 5.1, when the extreme residual dependence condition holds, the spectral measure ρ must satisfy the condition (33). The following theorem shows that all possible extreme value dependence structure and extreme residual dependence structure can occur.

Theorem 5.2. *Consider a continuous sample path stochastic process* $\{X(s)\}_{s\in[0,1]}$ *in the domain of attraction of a max-stable process with spectral measure* ρ*. Suppose* $\{X(s)\}_{s\in[0,1]}$ *is asymptotically jointly independent and exhibits extreme residual dependence characterized by an extreme residual coefficient* η *in* $(1/2, 1)$ *and a measure* Ψ^**. Then the spectral measure* ρ *must satisfy* (33)*, and the measure* Ψ^* *is finite on* $C_1^+[0,1]$*.*

Conversely, given a spectral measure ρ *satisfying* (33)*, a finite measure* Ψ^* *on* $C_1^+[0,1]$*, and any* $1/2 < \eta < 1$*, there exists a process* $\{X(s)\}_{s\in[0,1]}$ *belonging to the domain of attraction of a max-stable process with spectral measure* ρ*, exhibiting extreme residual dependence characterized by the given measure* Ψ^* *and having an extreme residual coefficient* η*.*

Proof of Theorem 5.2

The first half is proved by Theorem 5.1 and the construction of Ψ^*. It is only necessary to prove the inverse part. Similar to the two-dimensional case, we give a constructive proof. The proof is staged into three steps.

Firstly, we construct a stochastic process $\{W(s)\}_{s\in[0,1]}$ with extreme value dependence structure characterized by ρ. Although this has been proved in Theorem 9.4.1 in de Haan and Ferreira [2006], we provide our own construction which is necessary for the proof later.

Denote $c_0 = \rho(\bar{C}_1^+[0,1]) < \infty$. Then $\frac{1}{c_0}\rho$ is a probability distribution on $\bar{C}_1^+[0,1]$. Let Q_0 be a random element on $\bar{C}_1^+[0,1]$ following such a probability distribution. Let M_0 be a random variable independent from Q_0, with distribution function $P(M_0 > x) = \frac{c_0}{x}$, for all $x > c_0$. Then the constructed stochastic process W is given as

$$\{W(s)\}_{s\in[0,1]} := \{M_0 Q_0(s)\}_{s\in[0,1]} . \tag{38}$$

It is not difficult to verify that the marginal distributions of the stochastic process $\{W(s)\}_{s\in[0,1]}$ follows that

$$\lim_{t\to\infty} tP(\{W(s) > tx\}) = \frac{1}{x}. \tag{39}$$

Furthermore, for any $g \in C^+[0,1]$

$$\begin{aligned}\lim_{t\to\infty} tP(\{W \le tg\}^c) &= \lim_{t\to\infty} tP\left(\left\{M_0 > t \inf_{s\in[0,1]} \frac{g}{Q_0}\right\}\right)\\ &= E\frac{c_0}{\inf_{s\in[0,1]} \frac{g}{Q_0}}\\ &= \int_{\bar{C}_1^+[0,1]} \sup_{s\in[0,1]} \frac{f_1}{g} d\rho(f_1).\end{aligned} \tag{40}$$

Together with the marginal property (39), the process $\{W(s)\}_{s\in[0,1]}$ is in the domain of attraction of a max-stable process with spectral measure ρ.

Secondly, we construct a stochastic process that accommodates the extreme residual dependence characterized by Ψ^* and η as in the following proposition.

Proposition 5.1. *Given any finite measure Ψ^* on $C_1^+[0,1]$ and $1/2 < \eta < 1$, there exists a continuous sample path stochastic process $\{Z(s)\}_{s\in[0,1]}$ such that, for any $g \in C^+[0,1]$,*

$$\lim_{t\to\infty} tP(\{Z > tg\}) = \int_{C_1^+[0,1]} \frac{1}{\sup(f_2 g)} d\Psi^*(f_2) \tag{41}$$

$$\lim_{t\to\infty} t^\eta P(Z(s) > t) = 0, \quad \text{for } s \in [0,1]. \tag{42}$$

Proof of Proposition 5.1

The proof is similar to the two-dimensional case. Denote $c := \Psi^*(C_1^+[0,1]) < \infty$. Then $\frac{1}{c}\Psi^*$ is a probability distribution on $C_1^+[0,1]$. Consider a random element, Q in $C_1^+[0,1]$ following such a probability distribution. Let M be a random variable independent from Q, with distribution function $P(M > x) = \frac{c}{x}$, for all $x > c$. Moreover, since $1/2 < \eta < 1$, we take β such that $1 < \beta < 1/\eta$. Consider the stochastic process $\{Z(s)\}_{s\in[0,1]}$ given as

$$\{Z(s)\}_{s\in[0,1]} := \left\{ \frac{M}{Q(s)} \bigwedge M^\beta \right\}_{s\in[0,1]}.$$

It is clear that $\{Z(s)\}_{s\in[0,1]}$ is a continuous sample path process. We check condition (41) as follows. For any $g \in C^+[0,1]$,

$$\begin{aligned}
\lim_{t\to\infty} tP(\{Z > tg\}) &= \lim_{t\to\infty} tP\left(\{M > tQg\} \bigcap \left\{ M^\beta > t \sup_{s\in[0,1]} g(s) \right\}\right) \\
&= \lim_{t\to\infty} tP\left(\left\{ M > t\sup(Qg) \vee t^{1/\beta}(\sup g)^{1/\beta} \right\}\right) \\
&= \lim_{t\to\infty} tE_Q \frac{c}{t\sup(Qg) \vee t^{1/\beta}(\sup g)^{1/\beta}} \\
&= \lim_{t\to\infty} \int_{C_1^+[0,1]} \frac{1}{\sup(f_2 g) \vee t^{1/\beta - 1}(\sup g)^{1/\beta}} d\Psi^*(f_2) \\
&= \int_{C_1^+[0,1]} \frac{1}{\sup(f_2 g)} d\Psi^*(f_2).
\end{aligned}$$

The last equation comes from the Lebesgue dominated convergence theorem and the fact that the last integral is finite. The proof of relation (42) follows the same line as in the two-dimensional case. The proposition is thus proved.

The last step in our construction is to assemble the constructed $\{W(s)\}_{s\in[0,1]}$ and $\{Z(s)\}_{s\in[0,1]}$. Let

$$\{U(s)\}_{s\in[0,1]} := \left\{ W(s) \bigvee Z(s)^\eta \right\}_{s\in[0,1]},$$

where the processes $\{W(s)\}_{s\in[0,1]}$ and $\{Z(s)\}_{s\in[0,1]}$ are independent. We show that $\{U(s)\}_{s\in[0,1]}$ fulfills all requirements in Theorem 5.2. To achieve that, we check three relations:

a) by denoting the marginal distributions of $U(s)$ as $F_s(x)$, we have $\frac{1}{1-F_s(t)} \sim t$ as $t \to \infty$;
b) $\lim_{t\to\infty} tP\left(\{U < tg\}^c\right) = \int_{\bar{C}_1^+[0,1]} \sup_{s\in[0,1]} \frac{f_1}{g} d\rho(f_1)$, for all $g \in C^+[0,1]$;
c) $\lim_{t\to\infty} tP\left(\left\{U^{1/\eta} > tg\right\}\right) = \int_{\bar{C}_1^+[0,1]} \sup_{s\in[0,1]} \frac{1}{f_2 g} d\Psi^*(f_2)$, for all $g \in C^+[0,1]$.

a) Notice that for any $s \in [0,1]$, and $x > 0$,

$$\{U(s) > tx\} = \{W(s) > tx\} \cup \left\{Z(s) > (tx)^{1/\eta}\right\}.$$

Hence,

$$tP(W(s) > tx) \le tP(U(s) > tx) \le tP(W(s) > tx) + tP(Z(s) > (tx)^{1/\eta}).$$

Part **a)** is proved by the combination of (39) and (42).

b) For any $g \in C^+[0,1]$, we have that

$$\{U < tg\} = \{W < tg\} \cap \{Z^\eta < tg\}.$$

Thus,

$$\{U < tg\}^c = \{W < tg\}^c \cup \{Z^\eta < tg\}^c,$$

which implies that

$$tP\left(\{W < tg\}^c\right) \le tP\left(\{U < tg\}^c\right) \le tP\left(\{W < tg\}^c\right) + tP\left(\{Z^\eta < tg\}^c\right).$$

Considering (40), in order to prove **b)**, it is sufficient to prove that

$$\lim_{t\to\infty} tP\left(\{Z^\eta < tg\}^c\right) = 0.$$

By construction, $Z(s) \le M^\beta$ for all $s \in [0,1]$. Hence,

$$\{Z^\eta < tg\}^c \subset \left\{M^{\beta\eta} > t \inf_{s\in[0,1]} g\right\},$$

which implies that

$$tP\left(\{Z^\eta < tg\}^c\right) \le tP\left(\left\{M^{\beta\eta} > t \inf g\right\}\right) = O(t^{1-1/(\beta\eta)}).$$

From $\beta < 1/\eta$, we get that $1 - 1/(\beta\eta) < 0$. This completes part **b)**.

c) Given any $g \in C^+[0,1]$, on the one hand, we have that

$$\left\{U^{1/\eta} > tg\right\} \supset \{Z > tg\},$$

and on the other hand,

$$\begin{aligned}
&\left\{U^{1/\eta} > tg\right\} \\
\subset &\{Z > tg\} \cup \left\{W^{1/\eta} > tg\right\} \\
&\cup \left\{\exists s_1, s_2 \in [0,1], Z(s_1) > tg(s_1) \text{ and } (W(s_2))^{1/\eta} > tg(s_2)\right\} \\
:= &S_1 \cup S_2 \cup S_3.
\end{aligned}$$

Considering the relation (41), in order to prove **c)**, it is sufficient to prove that $\lim_{t\to\infty} tP(S_j) = 0$, for $j = 2, 3$.

Considering the restriction (33) on the spectral measure ρ, and the construction of $\{W(s)\}_{s\in[0,1]}$ in (38), we have $P(\inf_{s\in[0,1]} Q_0(s) > 0) = 0$, which implies that

$$P(\inf_{s\in[0,1]} W(s) > 0) = 0.$$

Because $S_2 \subset \left\{\inf_{s\in[0,1]} W(s) > 0\right\}$, we get $P(S_2) = 0$.

The last step is on S_3. By construction, $Z(s) \leq M^\beta$ and $W(s) \leq M_0$ hold for all $s \in [0, 1]$. The latter is implied by the fact that $\sup_{s\in[0,1]} Q_0(s) = 1$. Therefore, we have that

$$S_3 \subset \left\{M^\beta > t \inf g \text{ and } M_0^\eta > t \inf g\right\}.$$

Since M and M_0 are independent, we get that

$$\begin{aligned} tP(S_3) &\leq tP\left(M > (t \inf g)^{1/\beta}\right) P\left(M_0 > (t \inf g)^\eta\right) \\ &= tO(t^{-1/\beta})O(t^{-\eta}) \\ &= O(t^{1-1/\beta-\eta}). \end{aligned}$$

From $1 < \beta < 1/\eta$ and $1/2 < \eta < 1$, we get that $1 - 1/\beta - \eta < 1 - 2\eta < 0$. It is thus proved that $\lim_{t\to\infty} tP(S_3) = 0$, which completes the proof of part **c)**.

The combination of **a)** and **b)** implies that the constructed process $\{U(s)\}_{s\in[0,1]}$ belongs to the domain of attraction of a max-stable process with extreme value dependence structure characterized by the spectral measure ρ. The combination of **a)** and **c)** implies that $\{U(s)\}_{s\in[0,1]}$ has extreme residual dependence characterized by the measure Ψ^* with extreme residual coefficient η. Hence, Theorem 5.2 is proved.

Remark 5.2. *The extreme value dependence implies that* $\sup_{s\in[0,1]} \tilde{X}(s)$ *has an extreme value index 1, while the extreme residual dependence implies that* $\inf_{s\in[0,1]} \tilde{X}(s)$ *has an extreme value index* η*.*

6. Modeling systemic risk in banking system

It is often observed that banking crises are systemical, i.e. banks are likely to experience severe downside shocks simultaneously. This is called the systemic risk in banking system. One potential explanation of systemic risk is that banks share similar exposures to risk factors that are heavy-tailed, see, e.g. de Vries [2005]. This argument can be shown by a simple model as follows.

Consider a simple banking system with two banks (B_1, B_2) holding portfolio of risks on three independent risk factors C, L_1 and L_2. The losses of the two banks are given by

$$\begin{aligned} B_1 &= C + L_1 \\ B_2 &= C + L_2, \end{aligned} \tag{43}$$

where (C, L_1, L_2) indicates the losses generate by the risk factors. C is regarded as the common risk shared by B_1 and B_2, while L_1 and L_2 are idiosyncratic risks taken by individual banks respectively. Suppose (C, L_1, L_2) are independent and following heavy-tailed distributions. More specifically, we assume that L_i has tail index α and C has tail index β, i.e. as $x \to \infty$

$$\begin{array}{ll} P(L_i > x) & \sim \sigma_L x^{-\alpha}, \quad \text{for} \quad i = 1, 2, \\ P(C > x) & \sim \sigma_C x^{-\beta}, \end{array} \tag{44}$$

where $\sigma_L, \sigma_C > 0$ are the scales.

To examine the existence of the systemic risk, one may compare the probability of a systemic crisis with that of an individual crisis, i.e. calculating

$$\kappa := \lim_{t \to \infty} \frac{P(B_1 > t, B_2 > t)}{P(B_2 > t)}.$$

In de Vries [2005], it is shown that, when $\alpha = \beta$, $0 < \kappa < 1$. Hence the systemic risk exists, the banking system is fragile.

Deviating from the assumption $\alpha = \beta$ provides other possibilities on modeling different levels of systemic risk. The following theorem clarifies the different extreme dependence structure in different cases.

Theorem 6.1. *Consider a simple model on banking system in* (43) *and* (44). *Suppose* (C, L_1, L_2) *are all positive random variables. We have that*

a) *when* $\beta < \alpha$, (B_1, B_2) *are completely asymptotically dependent* ($\kappa = 1$);

b) *when* $\beta = \alpha$, (B_1, B_2) *are partially asymptotically dependent* ($0 < \kappa < 1$);

c) *when* $\alpha < \beta < 2\alpha$, (B_1, B_2) *are asymptotically independent* ($\kappa = 0$), *however, by denoting* $\tilde{B}_i = \frac{1}{1-F_i(B_i)}$ *where* F_i *is the marginal distribution of* B_i *for* $i = 1, 2$, *we have that*

$$\lim_{t \to \infty} t^{\beta/\alpha} P(\tilde{B}_1 > t, \tilde{B}_2 > t) = \sigma_L^{-\beta/\alpha} \sigma_C. \tag{45}$$

d) *when* $\beta > 2\alpha$, (B_1, B_2) *are asymptotically independent, moreover, with the same notation* $\tilde{B}_i$ *as in* **c)**,

$$\lim_{t \to \infty} t^2 P(\tilde{B}_1 > t, \tilde{B}_2 > t) = 1.$$

Proof of Theorem 6.1

The case **b)** is from de Vries [2005]. We only consider the other three cases.

a) $\beta < \alpha$

In this case, the common risk has a heavier tail than the idiosyncratic risks. From the Feller theorem, we have that as $t \to \infty$

$$P(B_i > t) \sim P(C > t).$$

On the other hand, for any $0 < \varepsilon < 1$, from

$$\{C > t\} \subset \{B_1 > t, B_2 > t\} \subset \{C > (1 - \varepsilon)t\} \cup \{L_1 > \varepsilon t, L_2 > \varepsilon t\}, \tag{46}$$

and the fact that $P(L_1 > \varepsilon t, L_2 > \varepsilon t) = o(P(C > t))$, we get that

$$1 \leq \liminf_{t\to\infty} \frac{P(B_1 > t, B_2 > t)}{P(C > t)} \leq \limsup_{t\to\infty} \frac{P(B_1 > t, B_2 > t)}{P(C > t)} \leq (1-\varepsilon)^{-\beta}.$$

By taking $\varepsilon \to 0$, we ge that, as $t \to \infty$,

$$P(B_1 > t, B_2 > t) \sim P(C > t) \sim P(B_2 > t).$$

Hence $\kappa = 1$.

c) $\alpha < \beta < 2\alpha$

In this case, as $t \to \infty$

$$P(B_i > t) \sim P(L_i > t) \sim \sigma_L t^{-\alpha}.$$

Hence, $\frac{1}{1-F_i(t)} \sim \frac{1}{\sigma_L} t^\alpha$ as $t \to \infty$.

Relation (46) implies that

$$0 \leq \frac{P(B_1 > t, B_2 > t)}{P(L_2 > t)} \leq \frac{P(C > (1-\varepsilon)t) + P(L_1 > \varepsilon t, L_2 > \varepsilon t)}{P(L_2 > t)}.$$

Since $\beta > \alpha$, $P(C > (1-\varepsilon)t) = o(P(L_2 > t))$. Thus, we get that

$$P(B_1 > t, B_2 > t) = o(P(L_2 > t)) = o(P(B_2 > t)),$$

i.e. $\kappa = 0$.

Next, from (46), notice that

$$P(L_1 > \varepsilon t, L_2 > \varepsilon t) = P(L_1 > \varepsilon t)P(L_2 > \varepsilon t) = O(t^{-\alpha})O(t^{-\alpha}) = o(t^{-\beta}) = o(P(C > t)).$$

Similar to case **b)**, we have that $P(B_1 > t, B_2 > t) \sim P(C > t)$, which implies that

$$\lim_{t\to\infty} t^\beta P(B_1 > t, B_2 > t) = \sigma_C.$$

From the marginal tail distribution of B_i, $\tilde{B}_i = \frac{1}{1-F_i(B_i)} \approx \frac{1}{\sigma_L} B_i^\alpha$ for large value of B_i. We have

$$\lim_{t\to\infty} t^{\beta/\alpha} P(\tilde{B}_1 > t, \tilde{B} > t) = \sigma_L^{-\beta/\alpha} \sigma_C.$$

d) $\beta > 2\alpha$

The proof of $\kappa = 0$ is similar to that in case **c)**. Analogous to the proof in **c)**, one can verify that

$$P(B_1 > t, B_2 > t) \sim P(L_1 > t)P(L_2 > t) \sim \sigma_L^2 t^{-2\alpha},$$

as $t \to \infty$. Together with the marginal tail distribution of B_i, it can be verified that

$$\lim_{t\to\infty} t^2 P(\tilde{B}_1 > t, \tilde{B}_2 > t) = 1.$$

Remark 6.1. *In theorem 6.1, we assume that C and L_i are all positive random variables for simplicity. Such an assumption is not essential: assuming that the left tails of C and L_i are lighter, i.e. having a higher tail indices, than the right tails, is sufficient for obtaining the same result.*

Remark 6.2. *When (B_1, B_2) are independent random variables, we have that*

$$\lim_{t\to\infty} t^2 P(\tilde{B}_1 > t, \tilde{B} > t) = \lim_{t\to\infty} tP(\tilde{B}_1 > t)tP(\tilde{B} > t) = 1.$$

Hence, the case **d)** *is comparable with the situation that systemic risk does not exist, while in the case* **c)**, *systemic risk still exists in the residual part.*

Remark 6.3. *In the case* **c)**, *relation (45) is comparable with (10) with $\eta = \alpha/\beta$. Due to the restriction that $\alpha < \beta < 2\alpha$, we have that $1/2 < \eta < 1$.*

To summarize, when the common risk factor dominates the risks taken by the banks, the systemic risk is at the same level as the individual risk, hence, the system is in the most fragile situation. When the common risk factor and the idiosyncratic risks have comparable tails, the systemic risk exists, but at a level proportional to the individual risk, hence, the system is in a less fragile situation. When the idiosyncratic risks dominates but the common risk is still considerably heavy, i.e. $\alpha < \beta < 2\alpha$, the system is in a least fragile situation, however, the systemic risk still exists in the residual part. When the common risk has a much lighter tail than that of the idiosyncratic risk, i.e. $\beta > 2\alpha$, the systemic risk does not exist.

Our extended model captures not only asymptotically dependent cases but also asymptotically independent cases with dependence in the residual parts. Although local financial institutions from different economic regions, bearing their idiosyncratic risks as their major risks, do not exhibit strong fragility, a global crisis may still lead to a systemic crash due to the dependence in the residual parts. This explains the phenomenon observed in a global crises: all financial institutions get a simultaneous systemic shock even though they may not be strongly linked. When modeling systemic risk within banking system, it is necessary to take into account dependence in residual parts, in order to avoid underestimation of systemic risk.

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